

Analysis of quantum semigroups with GKS–Lindblad generators: II. General

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 395303

(<http://iopscience.iop.org/1751-8121/41/39/395303>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.150

The article was downloaded on 03/06/2010 at 07:13

Please note that [terms and conditions apply](#).

Analysis of quantum semigroups with GKS–Lindblad generators: II. General

Bernhard Baumgartner and Heide Narnhofer

Fakultät für Physik, Universität Wien Boltzmanngasse 5, A-1090 Vienna, Austria

E-mail: Bernhard.Baumgartner@univie.ac.at and Heide.Narnhofer@univie.ac.at

Received 20 June 2008, in final form 30 July 2008

Published 2 September 2008

Online at stacks.iop.org/JPhysA/41/395303

Abstract

Semigroups describing the time evolution of open quantum systems in finite-dimensional spaces have generators of a special form, known as Lindblad generators. These generators and the corresponding processes of time evolution are analyzed and characterized as decay, dissipation and dephasing. In relation to these processes the Hilbert space of the system is equipped with a special structure, a decomposition into a sum of mutually orthogonal subspaces. The complete set of all the stationary states and the asymptotic behavior of the evolutions are presented in detail. Some unusual special facts about invariant operators and symmetries are studied and examples are demonstrated. Perturbation theory for the structure and for the stationary states is discussed and performed in case studies.

PACS numbers: 03.65.Yz, 05.40.–a, 42.50.Dv, 03.65.Fd

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Hundreds of papers dealing with ‘Lindblad equations’ have been written since their fundamental importance in the theory of open quantum systems was shown (see [L76, GKS76]). They give the proper mathematical form to Markovian semigroups of completely positive norm continuous maps which are needed in this context [D76]. The early investigations (see, for example [S76, F78, S80]) mostly aimed at establishing theories for the approach to thermal equilibrium, and all the subsequent investigations pursued special physical questions; none of them, however, dealt with the general mathematical structure. And this is now the theme of this paper. We give an analysis of the mathematical properties. It is analogous to the functional analysis of unitary groups of evolution, which was presented in the early phase of quantum mechanics. We continue the study on semigroups of completely positive maps, acting on states for systems with finite-dimensional Hilbert spaces, which we began in [BNT08].

While Schrödinger equations are studied in the Hilbert space spanned by the pure-state vectors, the Lindblad equations will be studied on two levels: one is the linear space of density matrices, the other one is the underlying Hilbert space. A relation between these two levels is established. In this relation we find a clear distinction between different processes: decay, dissipation and dephasing. Characterizing these processes involves a structuring of the underlying Hilbert space \mathcal{H} . Each Lindblad generator is related to a special decomposition of \mathcal{H} into mutually orthogonal subspaces. This decomposition can be seen as a generalization of the spectral decomposition of Schrödinger operators. Orthogonality of the subspaces holds in spite of non-Hermiticity of the defining operators.

We do not discuss ‘decoherence’. The meaning of this concept in physics is connected with the transition from quantumness to classicality, [J03, Z03a, Z03b], the mathematics of it requires the definition of a preferred basis. ‘Dephasing’ is a similar concept and we define what it means precisely in a mathematical fashion.

Our starting point, the connection with the earlier studies, is the result of [GKS76, L76].

Proposition 1 (generators of semigroups). *Every generator of a semigroup of completely positive trace preserving maps $T^t : \rho(s) \mapsto \rho(s+t)$ for $t \geq 0$ on the set of finite-dimensional density matrices ρ , can be written in the form*

$$\dot{\rho} = \mathcal{D}(\rho) = -i[H, \rho] + \sum_{\alpha} \mathcal{D}_{h_{\alpha}}(\rho) \tag{1}$$

where $H = H^{\dagger}$ is a Hamiltonian. Transition operators h_{α} define the irreversible parts, the simple generators:

$$\mathcal{D}_h(\rho) = h\rho h^{\dagger} - \frac{1}{2}(h^{\dagger}h\rho + \rho h^{\dagger}h). \tag{2}$$

This representation of \mathcal{D} is known as ‘diagonal’. For our purposes it is optimal. All that is needed to characterize the processes can be extracted from the set of the operators $\{H, h_{\alpha}, h_{\alpha}^{\dagger}\}$, and from the algebra of operators which commute with each element of this set. (A remark on the notation: we use here h_{α} with lower index, instead of the use of an upper index, which has been employed in our first paper [BNT08].) It is well known, [AF01, BP02], that the division of \mathcal{D} into a sum of several simple generators, attribution of $\{H, h_{\alpha}\}$ to \mathcal{D} , is not unique. Different sets of operators h_{α} can be attributed to a given generator \mathcal{D} , some actions can be shifted from the Hamiltonian H to the transfer operators h_{α} or in the other way. There exist attempts to favor special ways, like demanding $\text{Tr}[h_{\alpha}] = 0$, but we make here, in this paper, no restriction.

The mathematical analysis is of course related to special questions concerning physics:

- existence and characterization of stationary states;
- geometry of the paths and characterizing Hilbert space subspaces;
- aspects of symmetries;
- perturbation.

Identifying stationary states in \mathbf{S} , which is the set of all states of the system, is closely related to identifying special subspaces of the Hilbert space. The following general definitions and constructions are probably hard to swallow without examples. We present examples in section 5 and suggest having a look at them, parallel to reading definitions, propositions and theorems. Example 5.6 presents non-trivial asymptotics for example, using a representation of \mathcal{H} as a tensor product. (In some more general cases only a subspace of \mathcal{H} would be represented in this way.)

The summary of our investigations on the first two of the themes just mentioned is stated in theorems 2 and 3. These theorems treat the most general case. In special cases some of the

mentioned subspaces might be trivially empty, some numbers might be trivially one or zero. Again we suggest having a look at the examples to get an insight into the meaning of the many different concepts which are introduced in these two theorems. The deeper explanations will be presented in sections 2 and 3.

Theorem 2 (structuring of the Hilbert space).

- (1) *Decay: the Hilbert space can be represented in a unique way as a direct sum of two orthogonal subspaces, $\mathcal{H} = P_0\mathcal{H} \oplus P_0^\perp\mathcal{H}$, where $P_0^\perp\mathcal{H}$ is the maximal decaying subspace, i.e.*

$$\forall \rho \in \mathbf{S} : \lim_{t \rightarrow \infty} P_0^\perp T^t(\rho) P_0^\perp = 0, \quad (3)$$

and $P_0\mathcal{H}$ contains no decaying sub-subspace.

$$\forall Q \leq P_0 \exists \rho \in \mathbf{S} : \limsup_{t \rightarrow \infty} \text{Tr}[Q T^t(\rho) Q] \neq 0. \quad (4)$$

- (2) *Dephasing: the collecting subspace $P_0\mathcal{H}$ can further be divided in a unique way by splitting $P_0 = \sum_k Q_{0,k}$, $Q_{0,k} \cdot Q_{0,\ell} = \delta_{k\ell} Q_{0,k}$, into minimal subspaces $Q_{0,k}\mathcal{H}$ with relative dephasing*

$$\forall \rho(0), \quad \forall k, \ell, \quad k \neq \ell : \lim_{t \rightarrow \infty} Q_{0,k} \rho(t) Q_{0,\ell} = 0, \quad (5)$$

and the property, that the time evolution of each block $Q_{0,k} \rho(t) Q_{0,\ell}$ is independent of the other blocks.

- (3) *Asymptotics: each subspace $Q_{0,k}\mathcal{H}$ can be represented as a tensor product, $Q_{0,k}\mathcal{H} = \mathbb{C}^{n(k)} \otimes \mathcal{H}_{00,k}$, where $\dim(Q_{0,k}\mathcal{H}) = n(k) \cdot \dim(\mathcal{H}_{00,k})$, so that the time evolution at large times is described in block form, with Hamiltonians $H_{0,k} \otimes \mathbb{1}$ acting on $Q_{0,k}\mathcal{H}$ and with unique density matrices ρ_k supported by $\mathcal{H}_{00,k}$ inside each block:*

$$\forall \rho(0) \quad \exists \{\lambda_k, R_k\} : \lim_{t \rightarrow \infty} |\rho(t) - \bigoplus_k \lambda_k e^{-iH_{0,k}t} R_k e^{iH_{0,k}t} \otimes \rho_k| = 0, \quad (6)$$

where each R_k is a positive matrix with trace one acting on $\mathbb{C}^{n(k)}$, and $0 \leq \lambda_k \leq 1$, $\sum_k \lambda_k = 1$.

The set of stationary states is given by the set of density matrices

$$\bigoplus_k \lambda_k R_k \otimes \rho_k, \quad (7)$$

where $[H_{0,k}, R_k] = 0$, and ρ_k again unique for each k .

- (4) *Dissipation: the limiting density matrices inside the minimal blocks are of maximal rank: $\text{rank}(\rho_k) = \dim(\mathcal{H}_{00,k})$.*
- (5) *Cascades with basins: The decaying subspace $P_0^\perp\mathcal{H}$ can further be divided by splitting it as a cascade with ‘basins’ $P_{k,\ell}\mathcal{H}$, $k \geq 1$, all mutually orthogonal, $P_{k,\ell} \cdot P_{j,m} = \delta_{kj} \delta_{m\ell} P_{k,\ell}$, and arranged in levels $P_k\mathcal{H} = \bigoplus_\ell P_{k,\ell}\mathcal{H}$. So $P_0^\perp = \sum_{k \geq 1, \ell} P_{k,\ell}$, and the time evolution is like a ‘flow’, out of each basin into the ‘lower’ levels including collecting basins $P_{0,k}\mathcal{H} \subset P_0\mathcal{H}$, where each $Q_{0,\ell}\mathcal{H}$ mentioned in the item above contains $n(\ell)$ basins:*

$$\text{for } t > 0 : \quad P_j \cdot T^t(P_{k,\ell} \rho P_{k,\ell}) \cdot P_j \neq 0 \quad \text{if } j < k, \quad (8)$$

$$P_j \cdot T^t(P_{k,\ell} \rho P_{k,\ell}) \cdot P_j = 0 \quad \text{if } j > k, \quad (9)$$

$$P_{k,m} \cdot T^t(P_{k,\ell} \rho P_{k,\ell}) \cdot P_{k,m} = 0 \quad \text{if } m \neq \ell. \quad (10)$$

In special cases there is a possibility of unitarily reshuffling some basins, defining other basins $\tilde{P}_{k,\ell} = U \cdot P_{k,\ell} \cdot U^\dagger$. The number of minimal basins is unique, as are their dimensions.

All projectors P_k , $P_{k,\ell}$ and $Q_{0,k}$ are orthogonal projectors.

Perpendicular to the structuring into levels there are dissections into ‘enclosures’ with a division of density matrices into blocks with mutually independent evolutions.

Theorem 3 (enclosures and blocks). *If there exists a set of mutually orthogonal projectors Q_m , each commuting with H and every h_α , then the basins $P_{k,\ell}\mathcal{H}$ can be chosen in such a way that each subspace $Q_m\mathcal{H}$ is a direct sum of basins. We call such a $Q_m\mathcal{H}$ an enclosure. The time evolution of any density matrix ρ splits into mutually independent evolutions of blocks $Q_m\rho Q_\ell$:*

$$T^t(Q_m\rho Q_\ell) = Q_m T^t(\rho) Q_\ell. \tag{11}$$

As an example of the structuring, already well known, we can use the Grotrian diagram for helium. The dissection into enclosures is there the distinction between ortho- and parahelium. The lowest energy levels represent the collecting subspace. Energy levels with angular momentum zero are basins, those with higher angular momenta \vec{J} can be split further into basins, by diagonalizing a component of J . The choice of the axis of the component is not unique, but different kinds of splitting, with different basins, are unitarily equivalent.

The Q_m are projectors onto the ‘enclosures’—for details see section 3.1. In the Heisenberg picture, with time evolution of the observables, they are invariant. This is a conventional symmetry. But there are cases when other symmetries turn up in a non-conventional way, without conserved observables; cases when conservation of observables appears, without an overall symmetry; and the conserved observables need not commute with all the h_α , and they need not form an algebra. ‘Degeneracy’, the existence of several stationary states, may be connected with the occurrence of symmetries or not. The well-known connection of symmetries with conservation laws, fundamental for Langrangian mechanics and quantum mechanics, is here no longer valid. All this is discussed in section 4. It is also an important aspect when perturbations are studied, as is done in section 6.

2. Basic properties of the superoperators

2.1. The Hilbert–Schmidt space of matrices

Consider the set \mathbf{S} of states as the set of $n \times n$ density matrices $\rho_{i,j} = \langle i|\rho|j\rangle$, with $n = \dim(\mathcal{H})$, $\{|i\rangle\}$ some basis of \mathcal{H} , and consider it as embedded into the linear space of complex $n \times n$ matrices. Now consider the general mathematical wisdom on linear differential equations with constant coefficients, acting in a finite-dimensional space, which is here the space spanned by $n \times n$ -dimensional matrices. To each superoperator \mathcal{D} there exist eigenmatrices σ , proper or generalized:

$$\mathcal{D}(\sigma) = \lambda\sigma \quad \text{or} \quad (\mathcal{D} - \lambda)^n(\sigma) = 0.$$

So one has the special time evolutions: $\sigma(t) = e^{\lambda t} \cdot \text{polynomial}(t, \mathcal{D}^n(\sigma))$. The general solution to $\dot{\sigma} = \mathcal{D}(\sigma)$ is a linear combination of these special solutions. But the eigenmatrices to our \mathcal{D} may not be self-adjoint, although \mathcal{D} preserves self-adjointness. Instead, there may be pairs of eigenvalues $\{\lambda, \lambda^*\}$ with pairs of eigenmatrices $\{\sigma, \sigma^\dagger\}$. That is the reason why we do not stay in the space of self-adjoint matrices.

We endow this linear space with the Hilbert–Schmidt norm $\|\sigma\| = (\text{Tr}[\sigma^\dagger\sigma])^{1/2}$ and the inner product $\langle\langle\sigma|\rho\rangle\rangle = \text{Tr}[\sigma^\dagger\rho]$. The HS superspace of operators enables the definition of the adjoint of a super-operator. The adjoint super-operator \mathcal{D}^\dagger generates the evolution of observables in the Heisenberg picture, $\dot{F} = \mathcal{D}^\dagger(F)$. It acts as

$$\mathcal{D}^\dagger(F) = i[H, F] + \sum_\alpha \mathcal{D}_{h_\alpha}^\dagger(F), \tag{12}$$

where

$$\mathcal{D}_h^\dagger(F) = h^\dagger F h - \frac{1}{2}(h^\dagger h F + F h^\dagger h). \tag{13}$$

We remark that self-adjointness of \mathcal{D} is rather exceptional. See section 5 for examples.

2.2. Evolutions inside the set of states

T^t preserves self-adjointness, trace-norm and, for $t \geq 0$, positivity; it maps $\mathbf{S} \mapsto \mathbf{S}$, the set of normed positive matrices into itself.

The dissipative character of the evolution is reflected in the general changes of eigenvalues $r(t)$ of $\rho(t)$. (Here, and in the following, we write $\rho(t)$ for $T^t(\rho)$ and $\dot{\rho}(t)$ for $\frac{d}{dt}T^t(\rho)$.) Zero eigenvalues may become positive, positive eigenvalues may increase or decrease. But the decrease is bounded from below.

Proposition 4 (a differential inequality). *If $r(t)$ is a non-negative eigenvalue of $\rho(t)$, its change in time is bounded from below as*

$$\dot{r}(t) \geq - \left(\sum_{\alpha} \|h_{\alpha}\|^2 \right) r(t). \tag{14}$$

Proof. Differentiating the eigenvalue equation $\rho(t)\psi(t) = r(t)\psi(t)$, and using $\langle \psi | \dot{\psi} \rangle = 0$ gives

$$\begin{aligned} \dot{r}(t) &= \langle \psi | \dot{\rho} | \psi \rangle = \sum_{\alpha} \langle \psi | \left(h_{\alpha} \rho h_{\alpha}^\dagger - \frac{1}{2}(h_{\alpha}^\dagger h_{\alpha} \rho + \rho h_{\alpha}^\dagger h_{\alpha}) \right) | \psi \rangle \\ &\geq - \sum_{\alpha} \frac{1}{2} \langle \psi | (h_{\alpha}^\dagger h_{\alpha} \rho + \rho h_{\alpha}^\dagger h_{\alpha}) | \psi \rangle \\ &= - \sum_{\alpha} \langle \psi | h_{\alpha}^\dagger h_{\alpha} | \psi \rangle r(t). \end{aligned} \tag{15}$$

Here, the finite dimension of the Hilbert space is essential: it implies that the number of simple generators which are necessary to form some given \mathcal{D} is bounded by $\dim^2(\mathcal{H}) - 1$. All the sums over α are finite. □

An immediate consequence is that the positive eigenvalues r_j of the density operator obey the inequality

$$r_j(t) \geq \exp \left(- \sum_{\alpha} \|h_{\alpha}\|^2 t \right) \cdot r_j(0) > 0.$$

The rank of ρ cannot decrease in finite time, purification can only occur in the limit $t \rightarrow \infty$.

Moreover, we can affirm quite generally.

Proposition 5 (existence of stationary states). *For each \mathcal{D} there exists at least one stationary state $\rho \in \mathbf{S}$.*

Proof. The density matrices in $\mathcal{B}(\mathcal{H})$ form a compact convex set \mathbf{S} . We know that the semigroup maps \mathbf{S} into itself, and we may consider the map $\rho(0) \mapsto \rho(\Delta t)$ for some fixed time interval Δt . By the Brouwer fixed point theorem, there exists at least one fixed point,

$\rho(\Delta t) = \rho(0)$. So there exists the stationary state

$$\bar{\rho} = \frac{1}{\Delta t} \int_0^{\Delta t} \rho(t) dt, \quad (16)$$

satisfying $\mathcal{D}(\bar{\rho}) = 0$. \square

This abstract statement is needed in section 3.3. In the following it is then elucidated by more concrete formulae.

3. Processes and structuring

3.1. Definitions

Enclosure is not a standard concept, although it appears in Hamiltonian dynamics as a consequence of the conservation of energy. We define ‘enclosure’ as a subspace \mathcal{K} with the property that for all $\rho(0)$ the expectation of the orthogonal projector onto this subspace, $\text{Tr}[P_{\mathcal{K}}\rho(t)]$, is constant in time. In the Schrödinger picture this means that the system can neither leave nor enter this subspace. In the Heisenberg picture it means invariance of the observable $P_{\mathcal{K}}$ in the course of time. Each $P_{\mathcal{K}}$ is one of the projectors Q_m appearing in theorem 3. So, enclosure is a case of ‘conservation’, but in the special way that it is an orthogonal projector which is the conserved observable. In section 5 it is shown that there are cases where invariant observables exist, but no invariant projector, different to the evolutions under Hamiltonian dynamics.

Our analysis of general Lindblad equations \mathcal{T}^t starts with investigating an eventual enclosing of the system in the subspaces \mathcal{K} with conserved projectors $P_{\mathcal{K}}$. Then, *inside* the enclosures, decay and/or dissipation occurs. The phase relations between the enclosures may eventually show a dephasing. Dephasing is a typical quantum effect, but it is also related to *decoherence*, the transition from quantum to classical appearance [Z03a, Z03b]. We introduce precise mathematical definitions of these physical events of irreversible processes.

Definition 6 (irreversible events). *Decay occurs, if there exists a decaying subspace \mathcal{K} ; i.e.*

$$\forall \rho(0) : \text{Tr}[P_{\mathcal{K}}\rho(t)] \rightarrow 0. \quad (17)$$

Dissipation occurs, if there exists a subspace \mathcal{K} such that

$$\forall \rho(0) = P_{\mathcal{K}}\rho(0) : \text{rank}(\rho(t)) \rightarrow \dim(\mathcal{K}). \quad (18)$$

Dephasing occurs for phase relations, in the case of conservations of $P_{\mathcal{K}}$ and $P_{\mathcal{L}}$, with $\mathcal{K} \perp \mathcal{L}$, if

$$\forall \rho(0) : P_{\mathcal{K}}\rho(t)P_{\mathcal{L}} \rightarrow 0. \quad (19)$$

All arrows are meant as limit of t to infinity.

Enclosures, decaying and—their complement—collecting subspaces are the analog to the energy eigenspaces in Hamiltonian dynamics.

3.2. Enclosures and conservation of projectors

In the Heisenberg picture ‘conservation’ of P means $\mathcal{D}^\dagger(P) = 0$. In the Schrödinger picture this means that the state of the system can neither leave nor enter the subspace $P\mathcal{H}$.

Lemma 7. *Conservation of $P = P^\dagger = P^2$ is possible if and only if this observable commutes both with the Hamiltonian H and with all the transfer operators h_α ; in other words, iff it is an element of the von Neumann algebra $\{H, h_\alpha\}'$.*

Proof. We consider $\rho = P\rho P$, systems in the subspace $P\mathcal{H}$. Leaving this subspace is possible for the system, if

$$\exists \rho = P\rho P : P^\perp \mathcal{D}(\rho) P^\perp = \sum_\alpha P^\perp h_\alpha \rho h_\alpha^\dagger P^\perp \neq 0 \quad \Leftrightarrow \quad \exists \alpha : P^\perp h_\alpha P \neq 0.$$

Entering the same subspace means leaving the orthogonal complement and is thus possible for the system if $\exists \alpha : Ph_\alpha P^\perp \neq 0$. That neither leaving nor entering is possible implies therefore $\forall \alpha : Ph_\alpha = Ph_\alpha P = h_\alpha P$. Under this restriction one gets for the evolution of the phase relations, the off-diagonal part, $P\rho(t)P^\perp$:

$$P\mathcal{D}(\rho)P^\perp = -iP[H, \rho]P^\perp = iP\rho P H P^\perp,$$

which vanishes for every $\rho = P\rho P$ iff H commutes with P . And its vanishing is necessary, otherwise $P^\perp \mathcal{D}^2(\rho) P^\perp = P^\perp H P \rho P H P^\perp > 0$. \square

Assume that both P_i and P_j are conserved projectors. The commutations imply $\mathcal{D}(P_i \rho) = P_i \mathcal{D}(\rho)$ and $\mathcal{D}(\rho P_j) = \mathcal{D}(\rho) P_j$. So the evolution of each block $P_i \rho P_j$ is independent from the other parts of ρ . The set of conserved projectors generates the algebra $\mathcal{N} = \{H, h_\alpha\}'$. A maximal Abelian subalgebra of the von Neumann algebra \mathcal{N} gives a set of mutually orthogonal minimal conserved projectors P_i , a decomposition of \mathcal{H} and a decomposition of ρ as a block matrix, with split evolutions of the blocks. Such a decomposition is unique if \mathcal{N} is an Abelian algebra. If this algebra is not Abelian, then different decompositions are possible. They are related by unitary transformations with $U \in \mathcal{N}$ which leave \mathcal{D} invariant. This follows from the classification of finite-dimensional von Neumann algebras, stated for example in [J03b].

Conserved projectors come along with an algebra of conserved observables, $\{H, h_\alpha\}'$, and, moreover, with a dynamical symmetry:

$$T^t(U^\dagger \rho U) = U^\dagger \rho(t) U \quad \text{if } U \in \{H, h_\alpha\}'. \quad (20)$$

But there are cases of conserved observables which do not form an algebra; and on the other hand, conservation of projectors is not necessary for the occurrence of a dynamical symmetry. This is discussed in section 4.

A maximal set of mutually orthogonal minimal enclosing subspaces now enables the discussion of processes inside.

3.3. Collection into subspaces, dissipation inside

In studying the time evolution in [BNT08] we identified the condition ‘laziness’. A subspace $P\mathcal{H}$ is ‘lazy’ if there is no flow out of it in first order of time. This condition appears here as the first part for characterizing collecting spaces, the orthogonal complements of decaying spaces.

Definition 8 (lazy subspaces). *$P\mathcal{H}$ is a lazy subspace if*

$$\forall \rho = P\rho P : \frac{d}{dt} \text{Tr}[P\rho(t)P] = 0 \quad \text{at } t = 0. \quad (21)$$

Lemma 9. *$P\mathcal{H}$ is a lazy subspace \Leftrightarrow*

$$\forall \alpha : h_\alpha \cdot P = P \cdot h_\alpha \cdot P. \quad (22)$$

Proof. Since \mathcal{T}^t conserves the trace, one has $\text{Tr}[P\dot{\rho}(t)P] = -\text{Tr}[P^\perp\dot{\rho}(t)P^\perp]$. At time $t = 0$ this is to be expressed with the generating operator as

$$-\text{Tr}[P^\perp\mathcal{D}(\rho)P^\perp] = -\sum_{\alpha} \text{Tr}[P^\perp h_{\alpha} \rho h_{\alpha}^{\dagger} P^\perp] = -\sum_{\alpha} \text{Tr}[(P^\perp \cdot h_{\alpha} \cdot P)\rho(P \cdot h_{\alpha}^{\dagger} \cdot P^\perp)].$$

So, if $P^\perp \cdot h_{\alpha} \cdot P$ does not vanish for each α , there exists a state $\rho = P\rho P$, such that $\text{Tr}[P\dot{\rho}(t)P] \neq 0$. This inequality holds in particular for any $\rho = P\rho P$ which has $\text{rank}(\rho) = \text{Tr}[P]$. \square

Definition 10 (collecting subspaces). $P\mathcal{H}$ is a collecting subspace, if

$$\forall t > 0 \quad \forall \rho : \quad \mathcal{T}^t(P\rho P) = P\mathcal{T}^t(P\rho P)P. \quad (23)$$

Lemma 11. $P\mathcal{H}$ is a collecting subspace, iff it is a lazy subspace also satisfying

$$P\left(iH - \frac{1}{2}\sum_{\alpha} h_{\alpha}^{\dagger} h_{\alpha}\right)P^\perp = 0. \quad (24)$$

Proof. Since the Lindblad equation is of first order in taking the time derivative, the equation

$$\forall \rho : \quad \mathcal{D}(P\rho P) = P\mathcal{D}(P\rho P)P \quad (25)$$

is equivalent to (23). The laziness condition is equivalent to disabling, as is shown above, appearance of diagonal blocks $P^\perp\rho P^\perp$:

$$\forall \rho : \quad P^\perp\mathcal{D}(P\rho P)P^\perp = 0,$$

and (24) is equivalent to disabling appearance of off-diagonal blocks of $\rho(t)$,

$$\forall \rho : \quad P\mathcal{D}(P\rho P)P^\perp = 0.$$

This is seen just by inserting the Lindblad equation (1):

$$P\mathcal{D}(P\rho P)P^\perp = P\rho P\left(iH - \frac{1}{2}\sum_{\alpha} h_{\alpha}^{\dagger} h_{\alpha}\right)P^\perp. \quad (26) \quad \square$$

For the system there is no way (back) out of $P\mathcal{H}$. But some parts of ρ may enter this subspace, if P does not commute with the Hamiltonian H and all the transition operators h_{α} . In general $P \cdot \mathcal{T}^t(P^\perp\rho P^\perp) \cdot P \neq 0$. If P does not commute with all the h_{α} , this occurs for some ρ , for example $\rho = P^\perp/\text{Tr}[P^\perp]$, in first order in time, i.e. $P \cdot \mathcal{D}(P^\perp\rho P^\perp) \cdot P \neq 0$. If P does commute with all the h_{α} , but does not commute with the Hamiltonian, entering the subspace occurs in second order: $P \cdot \mathcal{D}^2(P^\perp\rho P^\perp) \cdot P \neq 0$; see the proof of lemma 7.

If P^\perp projects onto a decaying subspace, then $P\mathcal{H}$ is collecting. This can be seen in the structuring of \mathcal{H} , performed in section 3.4. For the simple $\mathcal{D} = \mathcal{D}_h$ there was no reason for such a detailed investigation: a collecting subspace is there just a proper eigenspace for the eigenvalue zero of h . But here the structures are in general richer. There is still an evolution going on in the collecting subspace. It can be seen as an evolution for density matrices defined on the reduced Hilbert space $P\mathcal{H}$.

$$\rho = P\rho P \quad \Rightarrow \quad \dot{\rho} = \check{\mathcal{D}}(\rho) \quad \text{defined with } \{\check{H} = PHP, \check{h}_{\alpha} = Ph_{\alpha}P\}. \quad (27)$$

If the collecting subspace is minimal, i.e. does not contain a smaller collecting subspace, the evolution inside is of the same type as in a minimal enclosure; it is dissipative, unless $\dim(P\mathcal{H}) = 1$.

The relations between characteristic subspaces and stationary states can now be analyzed, first for one direction of implications.

Lemma 12. *If ρ is a stationary state and P is the projector onto its range, then $P\mathcal{H}$ is a collecting subspace or an enclosure.*

Proof. This proof is done by revisiting the proofs for lemmas 9 and 11, with the special extra situation $\rho = P\rho P$ and $\text{range}(\rho) = P\mathcal{H}$. \square

The other relations state that inside a minimal enclosure, with no decay inside, there holds.

Theorem 13 (uniqueness of minimal stationary states). *Let \mathcal{K} be a subspace which is a minimal enclosure or a minimal collecting subspace, containing no smaller enclosure or collecting subspace. Then there exists one and only one stationary state supported by \mathcal{K} . Its density matrix has maximal rank, $\text{rank}(\rho) = \dim(\mathcal{K})$.*

Proof. There has to exist a stationary state ρ , see theorem 5, applied to the restricted evolution inside of \mathcal{K} . Linearity of the Lindblad equation implies that the stationary states, if more than one, form a line or a (hyper) plane. This would include elements in the boundary of \mathbf{S} with lower rank; but there are *no invariant states at the boundary of $\mathbf{S}(\mathcal{K})$* : any invariant state has a collecting subspace as its range (lemma 12), and it is assumed that \mathcal{K} contains no smaller collecting subspace. \square

3.4. Cascades of decay

In the cases where decay occurs it might be helpful or necessary to define a structure of the Hilbert space, analogous to the ‘energy levels’ in Grotrian diagrams of atoms.

We define the *lowest level of the cascade*, $P_0\mathcal{H}$, as the smallest subspace of \mathcal{H} which contains all minimal collecting subspaces. The strategy of further procedure to give more structuring is as follows: consider the complement of the lowest level, with the same evolution, except the flow out into the lowest level. This evolution, acting on $P_0^\perp\mathcal{H}$, is generated by \mathcal{D}_1 , formed with $P_0^\perp H P_0^\perp$ and $\{P_0^\perp h_\alpha P_0^\perp\}$. Then the first *higher level* $P_1\mathcal{H}$ of the cascade for \mathcal{D} is defined as the lowest level of the cascade for \mathcal{D}_1 . Iteration gives a series of levels $P_0\mathcal{H}, P_1\mathcal{H}, \dots, P_n\mathcal{H}$, until $\bigoplus_i P_i\mathcal{H} = \mathcal{H}$. Basins $P_{i,j}\mathcal{H}$ in the level $P_i\mathcal{H}$ are defined as *minimal* collecting subspaces for the reduced evolutions with no flow out into the lower levels.

It is a noticeable fact that two different basins are mutually orthogonal subspaces, unless there exists a symmetry between them, a unitary symmetry of the Lindblad equation. In such a case there is a larger set of equivalent basins, and there is a mutually orthogonal pair among them, spanning together the same space as is spanned by the non-orthogonal ones. The construction of an orthogonal partner basin is done in the third part of the proof of proposition 14 without any mentioning of symmetry. The proof that a symmetry has to exist in such a case, is in the proof of proposition 16.

The precise details:

Proposition 14 (decomposition into basins). *Each level $P_i\mathcal{H}$ can be decomposed into a direct sum of mutually orthogonal basins $P_{i,j}\mathcal{H}$. This decomposition is either unique or unique up to some unitary equivalence, which reshuffles basins among a set of partners with equivalent dissipations inside.*

Proof. Construct the decomposition inductively and begin with the lowest level.

- Consider $P\mathcal{H} := \bigoplus_j P_{0,j}\mathcal{H}$, a sum of minimal collecting mutually orthogonal subspaces, and consider another minimal collecting subspace $P_\sigma\mathcal{H}$, not contained in $P\mathcal{H}$. This $P_\sigma\mathcal{H}$ supports a unique stationary state σ (theorem 13).
- If $P_\sigma\mathcal{H}$ is orthogonal to $P\mathcal{H}$, define $P_{0,j+1} = P_\sigma$ and proceed inductively.
- If $P_\sigma\mathcal{H}$ is neither orthogonal to $P\mathcal{H}$ nor contained in it, consider the subspace

$$P_+\mathcal{H} := ((P\mathcal{H})^\perp \cap (P_\sigma\mathcal{H})^\perp)^\perp = (P + P_\sigma)\mathcal{H},$$

spanned by $P\mathcal{H}$ and $P_\sigma\mathcal{H}$. Note, that $P_+ \neq P + P_\sigma$; several steps to analyze the properties of $(P_+ - P)\mathcal{H}$ are needed. The sum of non-orthogonal collecting subspaces, $P_+\mathcal{H}$, is itself a collecting subspace. This follows from $P_+^\perp\mathcal{H} \subset P^\perp\mathcal{H}$ and $P_\sigma^\perp\mathcal{H} \subset P^\perp\mathcal{H}$. The formal implications are $P^\perp h_\alpha P = 0 \Rightarrow P_+^\perp h_\alpha P = 0$, and $P_\sigma^\perp h_\alpha P_\sigma = 0 \Rightarrow P_+^\perp h_\alpha P_\sigma = 0$, both together implying

$$(P + P_\sigma)^\perp h_\alpha P_+ = P_+^\perp h_\alpha P_+ = 0, \quad \text{for all } h_\alpha;$$

and, in the same way,

$$P \left(iH - \frac{1}{2} \sum_\alpha h_\alpha^\dagger h_\alpha \right) P^\perp = 0, \dots \Rightarrow P_+ \left(iH - \frac{1}{2} \sum_\alpha h_\alpha^\dagger h_\alpha \right) P_+^\perp = 0.$$

These equations are needed for applying lemmas 9 and 11.

The Lindblad equation in this collecting subspace $P_+\mathcal{H}$ can be described with restricted operators $\check{h}_\alpha = P_+ h_\alpha P_+$ and $\check{H} = P_+ H P_+$. We refer in the following to the block-matrix representation stated in the appendix. The conditions on $P\mathcal{H}$ of ‘laziness’ and being collecting, as stated in lemma 9 and in equation (24), are $C_\alpha = P^\perp \check{h}_\alpha P = 0$ and $iG - \frac{1}{2} \sum_\alpha A_\alpha^\dagger B_\alpha = P (i\check{H} - \frac{1}{2} \sum_\alpha \check{h}_\alpha^\dagger \check{h}_\alpha) P^\perp = 0$. With these two conditions, looking at the Lindblad equation acting on σ , resulting in $\dot{\sigma} = 0$, one gets for the block $S := P^\perp \sigma P^\perp$

$$\begin{aligned} \dot{S} = P^\perp \dot{\sigma} P^\perp &= -i[L, S] + \sum_\alpha \left(D_\alpha S D_\alpha^\dagger - \frac{1}{2} D_\alpha^\dagger D_\alpha S - \frac{1}{2} S D_\alpha^\dagger D_\alpha \right) \\ &\quad - \frac{1}{2} \sum_\alpha (B_\alpha^\dagger B_\alpha S + S B_\alpha^\dagger B_\alpha) = 0. \end{aligned}$$

Taking the trace gives $\sum_\alpha \text{Tr}[B_\alpha S B_\alpha^\dagger] = 0$. We note that $S \geq 0$, by positivity of σ , and, moreover, that this matrix has only strictly positive eigenvalues, $\text{rank}(S) = \text{rank}(P_+ - P)$, since $P_\sigma\mathcal{H}$ is minimal. This implies $\forall \alpha : B_\alpha = P \check{h}_\alpha P^\perp = 0$. Together with the ‘laziness’ condition $C_\alpha = 0$ this means commutation of every \check{h}_α with P , and this implies moreover, again using equation (24), that also \check{H} has to commute with P . It follows that

$$P_{0,j+1}\mathcal{H} := (P_+ - P)\mathcal{H}$$

is another collecting subspace, orthogonal to $P\mathcal{H}$.

- Iterating this procedure, until no other collecting subspace, no other stationary state not already contained in $P_0\mathcal{H} := P\mathcal{H}$ is left, gives a decomposition of the lowest level.
- Next, we investigate the restricted evolution of the complement of $P_0\mathcal{H}$. Consider the subspace $P_0^\perp\mathcal{H}$ and the processes generated by $\{P_0^\perp H P_0^\perp, P_0^\perp h_\alpha P_0^\perp\}$. The collecting subspaces of this evolution, with all outflow into the lowest level disabled, give the basins of the first higher level $P_1\mathcal{H} = \bigoplus_j P_{1,j}\mathcal{H}$. Then one iterates the disabling of the outflow of the remaining subspace, constructing the higher levels with basins $P_{i,j}\mathcal{H}$ until nothing more of \mathcal{H} remains.

The lowest level is the subspace spanned by all possible stationary states. So its definition is unique. By induction, the entire decomposition into levels is unique.

- Now consider two different decompositions of a level, w.l.o.g. of the lowest level. This gives exactly the situation treated above at the third point, with at least one stationary state σ with support in a basin $P_\sigma \mathcal{H}$ which is neither orthogonal to some basin $P_{0,j} \mathcal{H}$ nor contained in it. This is the case, iff there are some stationary phase relations $P_{0,j} \sigma P_{0,\ell}$, since σ is minimal and cannot be decomposed, $\sigma \neq P_{0,j} \sigma P_{0,j} + P_{0,j}^\perp \sigma P_{0,j}^\perp$. The discussion of such cases is postponed to section 3.5, using proposition 16. \square

The characterizations of basins involves a common Schur triangulation ([L69]) in block form of all the transfer operators. The blocks $P_{i,j} h_\alpha P_{k,\ell}$ with $i > k$ are zero. The change in time of a basin's content $P_{i,j} \rho P_{i,j}$ consists of

- inflow from (several) $P_{k,\ell} \mathcal{H}$ with $k > i$,
generated by $\{P_{i,j} H P_{k,\ell} + P_{k,\ell} H P_{i,j}, P_{i,j} h_\alpha P_{k,\ell}\}$
- outflow into (several) $P_{k,\ell} \mathcal{H}$ with $k < i$,
generated by $\{P_{i,j} H P_{k,\ell} + P_{k,\ell} H P_{i,j}, P_{k,\ell} h_\alpha P_{i,j}\}$
- dissipation inside the basin,
generated by $\{P_{i,j} H P_{i,j}, P_{i,j} h_\alpha P_{i,j}\}$.

If higher basins $P_{k,\ell} \mathcal{H}$ are empty, all contents of the basin $P_{i,j} \mathcal{H}$ will decay if $i \geq 1$. Dissipation leads to density matrices with full rank inside the basin, then, if $i \neq 0$, there is some outflow since $\sum_{k < i, \ell} P_{k,\ell} \cdot (iH + \frac{1}{2} \sum_\alpha h_\alpha^\dagger h_\alpha) \cdot P_{i,j} \neq 0$.

Proposition 15 (emptying of higher levels). *Only the lowest level in the cascade carries stationary states, formed by combinations of unique states inside each single basin; eventually there may be stationary phase relations. The collection of the higher levels in the cascade,*

$$\mathcal{K} = \bigoplus_{i \geq 1, j} P_{i,j} \mathcal{H} = P_0^\perp \mathcal{H},$$

is the maximal decaying subspace. It is completely emptied in the course of the evolution.

Proof. There are no stationary states in \mathcal{K} , by construction. So the diagonal blocks $P_{i,j} \rho P_{i,j}$ with $i \geq 1$ vanish. By preservation of positivity, the off-diagonal blocks $P_{i,j} \rho P_{k,\ell}$ and $P_{k,\ell} \rho P_{i,j}$ have to vanish also. \square

3.5. Dephasing and the geometry of paths

Here we study, as $t \rightarrow \infty$, the phase relations between basins. Knowing that basins in the upper levels of the cascade get empty, and phase relations involving one or two of the decaying basins have to vanish because of the preservation of positivity, it remains to study phase relations between minimal collecting subspaces. We may restrict the system and consider only the lowest level, $P_0 \mathcal{H}$, which is collecting. In this level the time evolution is identical to the evolution defined in equation (27), generated by the restricted operators. Every collecting subspace is there, in this restriction, an enclosure. So we simplify the discussion and consider a system with can be decomposed into minimal enclosures $P_i \mathcal{H}$ without decay. The time evolution of each block $P_i \rho P_j$ is independent of all the other blocks. When considering the phase relations between diagonal blocks we may therefore simplify further, and restrict the system to a space with just two basins $\mathcal{H} = P_i \mathcal{H} \oplus P_j \mathcal{H}$.

Proposition 16 (stationary phase relations). *A stationary phase relation $P_i \rho P_j$ exists if and only if there exists a unitary operator U commuting with H and each h_α which intertwines between the two enclosures.*

$$U P_j = P_i U, \quad U^2 = \mathbb{1}. \tag{28}$$

The stationary phase relation is unique up to a constant factor.

Proof. Assume that such an intertwiner U exists. It creates a dynamical symmetry, $\mathcal{D}(U\rho U^\dagger) = U\mathcal{D}(\rho)U^\dagger$, and it transforms the stationary density matrices of the enclosures into each other,

$$U\rho_j U^\dagger = \rho_i. \tag{29}$$

The commutation with the generating operators implies stationarity of phase relation blocks $P_i\rho P_j = r_{i,j}U\rho_j$ and $P_j\rho P_i = r_{j,i}\rho_j U$.

On the other hand, assume that some stationary block $P_i\rho P_j$ exists. Then also $P_j\rho P_i = (P_i\rho P_j)^\dagger$ is stationary. Since the matrices for ρ in diagonal blocks are of full rank (theorem 13), the density matrices

$$\sigma_\lambda = \frac{1}{2}(\rho_i + \lambda P_i\rho P_j + \lambda^* P_j\rho P_i + \rho_j)$$

are positive for $|\lambda|$ small. Now at some critical values of $|\lambda|$ the state σ_λ is at the boundary of \mathbf{S} , and $\text{rank}(\sigma_\lambda) < \text{rank}(P_i) + \text{rank}(P_j)$. This implies that $P_\lambda\mathcal{H}$, the range of σ_λ , is an enclosure (lemma 12), and that $P_\lambda \in \{H, h_\alpha\}'$. Since $P_i\mathcal{H}$ and $P_j\mathcal{H}$ are undecomposable and P_λ does neither commute with P_i nor with P_j , the only possibility for such a situation is, that the von Neumann algebra $\{H, h_\alpha\}'$ is not Abelian, and

$$\{H, h_\alpha\}' \cong \mathbb{C}^2 \otimes P_i\mathcal{H},$$

including a unitary intertwiner U acting as in equation (29). □

Proof (uniqueness of decomposition into basins). Proposition 16 gives the completion for proving proposition 14. The decomposition into basins is not unique, iff there exist stationary phase relations. And such stationary phase relations exist, iff there is a unitary equivalence as stated in proposition 16. □

The geometry of paths $\{\rho(t)\} \subset \mathbf{S}$ is related to the eigenvalues of the superoperator \mathcal{D} . Each path can be decomposed into a sum of at least one stationary state and paths of self-adjoint matrices which are eigenmatrices or pairwise sums of eigenmatrices of \mathcal{D} .

Special paths for self-adjoint matrices:

- eigenvalue zero \Leftrightarrow stationary state;
- imaginary eigenvalue \Leftrightarrow circular path;
- negative eigenvalue \Leftrightarrow path leading straight to zero;
- complex pairs of eigenvalues with THE negative real part \Leftrightarrow paths formed as $\gamma\sigma(t) + \gamma^*\sigma^\dagger(t)$, spiraling into zero.

General paths arise as superpositions of special paths. Geometric considerations give some implications for the eigenvalues of \mathcal{D} : decay processes lead straight to the boundary of \mathbf{S} , so negative eigenvalues are involved. Eigenmatrices whose range contains some part of $P_0^\perp\mathcal{H}$ belong to eigenvalues with the negative real part. The eigenmatrices whose support is in $P_0\mathcal{H}$ can be chosen such that their supports are in single collecting basins or in blocks giving phase relations between pairs of basins. In this way one gets a complete set of eigenmatrices spanning the space of all matrices with support in $P_0\mathcal{H}$. Since no path can leave \mathbf{S} , there are no eigenvalues with the positive real part. Moreover we can state

Lemma 17. *To the eigenvalue zero of \mathcal{D} there exist only proper eigenmatrices. The corresponding eigenspace is spanned by positive density matrices.*

Proof. Consider the stationary eigenmatrix σ . If it is not self-adjoint, then also σ^\dagger is an eigenmatrix, as are the self-adjoint $\sigma + \sigma^\dagger$ and $i\sigma - i\sigma^\dagger$. If σ is self-adjoint but not positive, consider it split as $\sigma = \rho_+ - \rho_-$, both parts being positive. Since T^t is positivity

preserving, both parts separately must be stationary. Now assume the existence of a generalized eigenmatrix τ_0 , with $\mathcal{D}(\tau_0) = \sigma$, $\mathcal{D}(\sigma) = 0$. Again one can conclude that the analog equations should hold for the adjoint matrices and for their linear combinations. One can therefore assume $\tau_0 = \tau_0^\dagger$. Integrating the evolution equation, assuming $\tau(0) = \tau_0$, gives $\tau(t) = \tau_0 + t\sigma$. Multiplying with some small ε and adding some positive ρ with full rank would give a path starting inside \mathbf{S} but leaving it as t gets large. This is a contradiction to the preservation of positivity, so no such τ_0 can exist. \square

The eigenvalues of \mathcal{D} lying on the imaginary axis correspond to circular paths. Such cases can appear for phase relations, for off-diagonal blocks of ρ .

Theorem 18 (Dephasing and non-dephasing; eigenvalues of \mathcal{D}).

- (1) For a minimal block at the diagonal, belonging to a collecting basin, $\{\sigma = P_{0,j}\sigma P_{0,j}\}$, there exists exactly one eigenmatrix to the eigenvalue zero. All other eigenvalues have the negative real part.
- (2) For an off-diagonal block $\{\sigma = P_{0,j}\sigma P_{0,k}\}$ where there exists an intertwiner $U P_j = P_i U$ with $U \in \{P_0 H P_0, P_0 h_\alpha P_0\}'$, there exists exactly one eigenmatrix to the eigenvalue zero. The eigenmatrix is $U \rho_j$, where ρ_j is the stationary eigenmatrix with support in $P_{0,j} \mathcal{H}$. All other eigenvalues have the negative real part.
- (3) For an off-diagonal block $\{\sigma = P_{0,j}\sigma P_{0,k}\}$ where there exists an intertwiner $U \in \{P_0 H P_0 - (E_j P_{0,j} + E_\ell P_{0,\ell}), P_0 h_\alpha P_0\}'$, there exists exactly one eigenmatrix to an eigenvalue on the axis of imaginary numbers. The eigenvalue is $i(E_\ell - E_j)$, the eigenmatrix is $U \rho_j$, where ρ_j is the stationary eigenmatrix with support in $P_{0,j} \mathcal{H}$. All other eigenvalues have the negative real part.
- (4) For an off-diagonal block $\{\sigma = P_{0,j}\sigma P_{0,k}\}$ where there is no intertwiner as in item (2) or (3) there exist only eigenvalues with the negative real part.

Proof. The existence and uniqueness of an eigenmatrix to the eigenvalue zero in cases (1), (2), and the nonexistence in case (4) are stated and then proven in theorem 13 and in proposition 16, then lemma 17 states that there are no generalized eigenspaces to this eigenvalue.

It remains to examine the existence or non-existence of other eigenvalues on the imaginary axis. The method is the same as in the proof of theorem 17 ‘No circular paths’ in [BNT08]. We switch between the Schrödinger and the Heisenberg picture. \mathcal{D}^\dagger has the same spectrum as \mathcal{D} . Assume the existence of an eigenvalue $\lambda = ir$ with eigenoperator $F = P_{0,\ell} F P_{0,j}$, where $r \in \mathbb{R}$, so $\exp(t\mathcal{D}^\dagger)F = e^{ir t} F$, and use the Kadison inequality. No further details for the time evolution are needed to deduce equation (30). We refer to [BNT08] for description of how to conclude that $V := F/\sqrt{\|F^\dagger F\|}$ is a local isometry between $P_{0,j} \mathcal{H}$ and $P_{0,\ell} \mathcal{H}$ or a local unitary if $j = \ell$:

$$V^\dagger \cdot V = P_{0,j}, \quad V \cdot V^\dagger = P_{0,\ell}. \tag{30}$$

Now we use the evolution equation (12) for V , multiply from the left by V^\dagger , and get

$$V^\dagger \mathcal{D}^\dagger(V) = i(V^\dagger H V - P_{0,j} H) + \sum_\alpha \left(V^\dagger h_\alpha^\dagger V h_\alpha - \frac{1}{2} P_{0,j} h_\alpha^\dagger h_\alpha - \frac{1}{2} V^\dagger h_\alpha^\dagger h_\alpha V \right) = ir P_{0,j}. \tag{31}$$

Since, by definition of the collecting basins, $\{P_{0,j}, P_{0,\ell}\} \subset \{P_0 H P_0, P_0 h_\alpha P_0\}'$, we may define

$$\check{h}_\alpha = P_{0,j} h_\alpha = h_\alpha P_{0,j}, \quad \hat{h}_\alpha = V^\dagger \check{h}_\alpha V, \quad \check{H} = P_{0,j} H, \quad \hat{H} = V^\dagger \check{H} V$$

and write the trace of (31) as

$$i\text{Tr}[\hat{H} - \check{H}] + \sum_{\alpha} \left(\text{Tr}[\hat{h}_{\alpha}^{\dagger} \check{h}_{\alpha}] - \frac{1}{2} \text{Tr}[\check{h}_{\alpha}^{\dagger} \check{h}_{\alpha}] - \frac{1}{2} \text{Tr}[\hat{h}_{\alpha}^{\dagger} \hat{h}_{\alpha}] \right) = i r \dim(P_{0,j} H). \quad (32)$$

The Cauchy–Schwarz inequality, applied to inner product $\text{Tr}[\cdot \cdot]$ in the H.S. space of matrices, and the inequality between geometric and arithmetic mean imply that the real part of the lhs of (32) is zero iff $\forall \alpha : \hat{h}_{\alpha} = \check{h}_{\alpha}$, i.e. $V^{\dagger} h_{\alpha} V = P_{0,j} h_{\alpha}$, $h_{\alpha} V = V h_{\alpha}$. Using again (31) this implies $H \cdot V = V \cdot (H + r)$.

There are no such imaginary eigenvalues, no circular paths, in cases (1), (2) and (4). They exist only in case (3), with $U = V + V^{\dagger} + \bigoplus_{k \neq j, k \neq \ell} P_{0,k}$. \square

3.6. Stationary states, collection of results

A collection of results stated above in this section now gives the proofs of the main theorems.

Proof. For theorem 2 on structuring of the Hilbert space: in proposition 14 on decomposition into basins the decomposition of \mathcal{H} into $P_0 \mathcal{H}$, defined as the lowest level of the cascade of decay, and its complement $P_0^{\dagger} \mathcal{H}$ is performed. This gives part (1) of the theorem.

Proposition 14 also gives the further splitting of P_0 into basins. These are minimal collecting subspaces. If there exist stationary phase relations between two basins, and only then, can this splitting be varied, using another pair of basins which are related to the former two and among themselves by unitary transformations. This is stated and then proven in proposition 16 on stationary phase relations. Their existence comes with a form of equivalence between basins, given by the unitary transformation, which has to commute with every $P_0 h_{\alpha} P_0$ and with $P_0 H P_0$.

There may exist a generalized form of equivalence between basins, where the unitary transformation commutes again with every $P_0 h_{\alpha} P_0$ but then with $P_0 H P_0 - E_j P_{0,j} + E_{\ell} P_{0,\ell}$ instead of $P_0 H P_0$. Such a generalized form of equivalence implies again equivalence of the unique stationary states located in the basins; and it enables the occurrence of undamped oscillating phase relations. Collecting equivalent and generalized equivalent basins into subspaces $Q_{0,k} \mathcal{H}$ gives the larger, unique, part of splitting $P_0 = \sum_k Q_{0,k}$. There are neither stationary nor undamped oscillating phase relations $Q_{0,k} \mathcal{H} Q_{0,\ell}$. This is stated and then proven in theorem 18 on dephasing and non-dephasing.

The recollection of basins can be seen as $Q_{0,k} \mathcal{H} = \mathbb{C}^n(k) \otimes \mathcal{H}_{00,k}$. Uniqueness of the stationary states ρ_k in the minimal collecting subspaces, everyone equivalent to $\mathcal{H}_{00,k}$, is stated in theorem 13. So parts (2)–(4) of the theorem are proven.

The splitting of $P_0^{\dagger} \mathcal{H}$ into higher levels and basins is stated in proposition 16, proving the last part of the theorem. \square

A subspace \mathcal{K} is the minimal support of a minimal stationary state iff it fulfils the conditions

- ‘Laziness’—the projector $P_{\mathcal{K}}$ fulfils equation (22);
- ‘No creation of off-diagonal elements’— $P_{\mathcal{K}}$ fulfils equation (24);
- ‘Minimality’— \mathcal{K} contains no smaller subspace fulfilling the first two conditions.

The characteristic equations have also been found by Kraus *et al* [K08a, K08b], as determining ‘dark states’, which are pure stationary states.

A collection of results gives also the

Proof. For theorem 3 on enclosures and blocks: the relations of projectors $Q_m \in \{\mathcal{H}, h_{\alpha}\}'$ to the time evolution and to ‘enclosure’ are analyzed in lemma 7 and in the further discussions

in section 3.2 on ‘Conservation of projectors and enclosures’. Because of the enclosure, all further decompositions can be performed for the restricted evolution acting on the set of density matrices ρ with support in $Q_m \mathcal{H}$. \square

4. Invariance (conservation) and symmetry

4.1. Invariant observables

The stationary states span a subspace of the HS space of matrices, the space of eigenmatrices to the eigenvalue zero of the superoperator \mathcal{D} . Its adjoint, \mathcal{D}^\dagger , has the same set of eigenvalues, so there is a linear set of invariant operators. It has the same dimension as the set of stationary states, and contains only proper ‘eigenmatrices’, in duality to lemma 17.

Let us start a construction of an invariant observable with a nucleus, located in a collecting basin, say $P_{0,k} \mathcal{H}$. The defining condition (23) ‘nothing goes out’ is equivalent to the dual condition ‘no observable comes in’

$$P_{0,k} \mathcal{T}^{t\dagger}(F) P_{0,k} = P_{0,k} \mathcal{T}^{t\dagger}(P_{0,k} F P_{0,k}) P_{0,k}. \quad (33)$$

Inside the basin is full dissipation with only one stationary state. There is therefore only one invariant observable inside the basin (up to constant factors). It is the projector $P_{0,k}$.

$$P_{0,k} \mathcal{T}^{t\dagger}(P_{0,k}) P_{0,k} = P_{0,k}. \quad (34)$$

The Heisenberg evolution goes backward. It lets, applied to the projector $P_{0,k}$, the diagonal block $P_{0,k}$ unchanged, it creates an extension into the decaying subspace P_0^\perp , and also phase relations between this subspace and the basin where it started. It lets the whole block $P_0 \mathcal{H}$ unchanged, since this collecting subspace is spanned by collecting basins, each one showing the ‘no observable comes in’ condition (33). The evolved observable therefore stays HS orthogonal to eventually existing undamped oscillating phase relations, and we can define

$$A_{0,k} := \lim_{t \rightarrow \infty} \mathcal{T}^{t\dagger}(P_{0,k}) \quad (35)$$

as an invariant observable. It is positive, since preserving positivity goes over from \mathcal{T}^t to $\mathcal{T}^{t\dagger}$ by duality.

For a maximal set of mutually orthogonal collecting basins one obtains

$$\sum_k P_{0,k} = P_0 \iff \sum_k A_{0,k} = \mathbb{1}. \quad (36)$$

There may be a still larger set of linearly independent invariant observables. Let us represent subspaces with n equivalent collecting basins $P_{0,\ell}$ allowing for stationary phase relations, but not undamped oscillating ones, as

$$\bigoplus_{\ell} P_{0,\ell} = \mathbb{C}^n \otimes \mathcal{H}_{00}.$$

For any $n \times n$ matrix M the observable $M \otimes \mathbb{1}$ can be extended, analogously to procedure (35), to an invariant observable. n^2 of these observables can be linearly independent, n of them are as constructed in (35).

An algebraic analysis of the set of stationary states, of invariant observables and their relations to the generating operators relies on first cutting off the decaying subspace $P_0^\perp \mathcal{H}$. The restricted time evolution, generated by \mathcal{D} defined with Hamiltonian and transfer operators $\{P_0 H P_0, P_0 h_\alpha P_0\}$, is identical to the full one for ρ with support on $P_0 \mathcal{H}$. For this restricted system, on $P_0 \mathcal{H}$, the concepts of ‘basin’ and ‘enclosure’ are identical. Projectors onto basins are elements of the von Neumann algebra $\mathcal{N}_0 = \{P_0 H P_0, P_0 h_\alpha P_0\}'$, which is the set of invariant

observables for the restricted time evolution. The set of extended invariant observables in the large system is thus related to the commuting von Neumann algebra of the restricted system; restricted to the non-decaying level $P_0\mathcal{H}$.

$$\exists F : \mathcal{D}^\dagger(F) = \dot{F} = 0 \iff P_0 F P_0 \in \mathcal{N}_0 \quad (37)$$

So, in systems without decay, the invariant observables do form an algebra. But, in systems with decay, extension (35) does in general destroy this property. See examples in section 5.

4.2. Symmetries

A dynamical symmetry is defined by the existence of unitary or anti-unitary operators V , such that

$$\forall t, \forall \rho \quad T^t(V\rho V^{-1}) = V\rho(t)V^{-1}. \quad (38)$$

This is equivalent to

$$\forall \rho \quad \mathcal{D}(V\rho V^{-1}) = V\mathcal{D}(\rho)V^{-1}. \quad (39)$$

Sometimes an appearance of a dynamical symmetry is in connection with the existence of an algebra of conserved observables, a connection well known in Hamiltonian dynamics. If $U \in \{H, h_\alpha\}'$ then $T^t(U\rho U^\dagger) = U\rho(t)U^\dagger$. Another way how a symmetry may be guaranteed is, that $VHV^{-1} = H$ and the set $\{Vh_\alpha V^{-1}\}$ equals the set $\{h_\alpha\}$. But it may also be hidden, not immediately to be observed in the h_α . See examples in section 5. For continuous groups, and for \mathcal{D} with a finite number of transfer operators h_α , the symmetry is necessarily not completely represented by invariance of the set of generators.

The *maximal* symmetry is invariance under *all* unitary and anti-unitary transformations. There is only one ray of generators compatible with this symmetry, in the center of the *cone of generators*, see [BNT08]. It consists of $\{\mathcal{D}|\mathcal{D}(\rho) = \lambda \cdot (\omega - \rho)\}$, where $\omega = \mathbb{1}/\dim(\mathcal{H})$ is the completely mixed state. There are several ways of choosing the set $\{h_\alpha\}$ to form such a special \mathcal{D} . Examples are again in section 5.

While, in Hamiltonian dynamics, the appearances of

- dynamical symmetry,
- invariant (conserved) observables,
- algebra formed by the invariant observables,

must come together, these relations are not strict in irreversible dynamics. Here we observe cases of

- dynamical symmetry without invariant observables,
- invariance of observables without a symmetry,
- invariant observables which do not form an algebra.

For each of these cases we present examples.

There remains a relation between a dynamical symmetry and the set of stationary states. If V is a symmetry operator as in (38), then the set of stationary states obeys the *symmetry for stationarity*

$$\{\rho \mid \text{stationary}\} = \{V\rho V^{-1} \mid \rho \text{ stationary}\}. \quad (40)$$

But this is, in general, a one-way relation. The symmetry for stationarity (40) may be valid, without (38) being true. This comes, again, because of the restriction of the set of invariant states to the subspace $P_0\mathcal{H}$.

5. Examples

The density matrices are representations of states in some bases of the usual type, employing a complete orthogonal set of basis vectors. Mostly we use \mathcal{D} given as a sum of two simple generators, with transition operators $\{h_+, h_-\}$. Zeroes as matrix elements are represented with dots. Lower indices on \mathbb{C} indicate the role of a subspace as a level or as a basin. Matrix elements of the density operator are denoted as $r_{i,j}$. Representations using a tensor product are in accordance with its use in part (3) of theorem 2. ‘No invariant observable’ means, precisely: only the constants are invariant.

5.1. Dissipation

Hilbert space $\mathcal{H} = \mathbb{C}^2$

$$h_+ = \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} \quad h_- = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$$

There is a unique stationary state. It is $\omega = \mathbb{1}/2$, a fact which cannot occur with simple generators. To represent the dynamics we use Pauli matrices, so $h_{\pm} = \sigma_{\pm} = (\sigma_x + \sigma_y)/2$ and $\mathcal{D}(\sigma_{\pm}) = -\sigma_{\pm}$, $\mathcal{D}(\sigma_z) = -2\sigma_z$. The dynamics is symmetric under rotation around the z -axis, under reflection $\sigma_z \leftrightarrow -\sigma_z$ and under complex conjugation C . The symmetry for stationarity is maximal, but there is no invariant observable.

5.2. Decay, two collecting basins, no stationary phase relations

$\mathcal{H} = \mathbb{C}^3 = \mathbb{C}_{0,1} \oplus \mathbb{C}_{0,2} \oplus \mathbb{C}_1$

$$h_+ = \begin{pmatrix} 1 & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad h_- = \begin{pmatrix} 1 & \cdot & -1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

Extremal stationary states are $1 \oplus 0 \oplus 0$ and $0 \oplus 1 \oplus 0$. The symmetry for stationarity includes exchange of these extremal states and complex conjugation. The whole system is only one enclosure. Invariant observables are diagonal matrices with components $(1, 0, 2/3)$, or $(0, 1, 1/3)$, and their linear combinations. These invariant observables do not form an algebra. There exists no dynamical symmetry but complete conjugation C .

5.3. Decay, two collecting basins with stationary phase relations

$\mathcal{H} = \mathbb{C}^3 = \mathbb{C}_{0,1} \oplus \mathbb{C}_{0,2} \oplus \mathbb{C}_1$

$$h_+ = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad h_- = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

Stationary states: any 2×2 density matrix with support on $\mathbb{C}_{0,1} \oplus \mathbb{C}_{0,2}$. Symmetry for stationarity: $U \oplus \mathbb{1}$ with any unitary U , and C . No dynamical symmetry but C . Invariant observables: linear combinations of A_1, \dots, A_4 ,

$$A_1 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 2/3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1/3 \end{pmatrix}, \quad A_3 = A_4^\dagger = \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1/3 \end{pmatrix}.$$

They do not form an algebra. See figure 1.

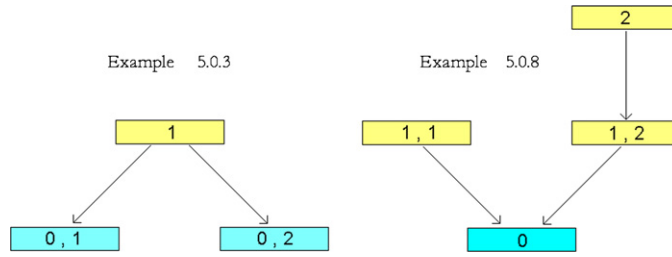


Figure 1. Two cascades of decay. Basins are represented by rectangles, decaying as yellow, collecting as blue. Each flow between two basins is indicated by an arrow.

5.4. Decay, basins with dissipation inside

$$\mathcal{H} = \mathbb{C}^4 = \mathbb{C}_{0,1}^2 \oplus \mathbb{C}_{0,2}^2$$

$$h_{\pm} = \begin{pmatrix} \cdot & \pm 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \pm 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}.$$

The restricted evolution on $\mathbb{C}_{0,1}^2$ is almost the same as in section 5.1, only with twice the speed. One stationary state, $\omega \oplus 0$, no invariant observables. Symmetry for stationarity: all the unitary and anti-unitary transformations acting on $\mathbb{C}_{0,1}^2$. No invariant observables.

5.5. Dephasing of two enclosures

$$\mathcal{H} = \mathbb{C}^4 = \mathbb{C}_{0,1}^2 \oplus \mathbb{C}_{0,2}^2$$

$$h_{+} = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad h_{-} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \end{pmatrix}.$$

The restricted evolutions on each $\mathbb{C}_{0,j}^2$ is the same as in section 5.1. Acting on the off-diagonal blocks, these evolutions are ‘out of phase’, destroying every phase relation. Such an off-diagonal block evolves according to

$$\frac{d}{dt} \begin{pmatrix} r_{1,3} & r_{1,4} \\ r_{2,3} & r_{2,4} \end{pmatrix} = \begin{pmatrix} r_{2,4} - r_{1,3} & -r_{1,4} \\ -r_{2,3} & -r_{1,3} - r_{2,4} \end{pmatrix}.$$

Two extremal stationary states $\omega \oplus 0$ and $0 \oplus \omega$, two invariant observables, $\mathbb{1} \oplus 0$ and $0 \oplus \mathbb{1}$ are very rich symmetry.

5.6. Undamped oscillating phase relation

$$\mathcal{H} = \mathbb{C}^4 = \mathbb{C}_{0,1}^2 \oplus \mathbb{C}_{0,2}^2 \cong \mathbb{C}^2 \otimes \mathbb{C}_{0,0}^2$$

$$H = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad h_{+} = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad h_{-} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}.$$

The evolutions of states of the enclosures $\mathbb{C}_{0,j}^2$ are again as in section 5.1. Here they are in phase when acting on the off-diagonal blocks, leaving a special phase relation undamped. But the Hamiltonian creates an oscillation.

$$\frac{d}{dt} \begin{pmatrix} r_{1,3} & r_{1,4} \\ r_{2,3} & r_{2,4} \end{pmatrix} = -i \begin{pmatrix} r_{1,3} & r_{1,4} \\ r_{2,3} & r_{2,4} \end{pmatrix} + \begin{pmatrix} r_{2,4} - r_{1,3} & -r_{1,4} \\ -r_{2,3} & r_{1,3} - r_{2,4} \end{pmatrix}.$$

So

$$\mathcal{T}^t \begin{pmatrix} r_{1,3} & r_{1,4} \\ r_{2,3} & r_{2,4} \end{pmatrix} \sim_{t \rightarrow \infty} \sim e^{-it \frac{r_{1,3} + r_{2,4}}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The equation for full dynamics can be written as

$$\frac{d}{dt} (\mathbf{M} \otimes \rho) = -i[H, \mathbf{M}] \otimes \rho + \mathbf{M} \otimes \mathcal{D}_{0,0}(\rho),$$

with $\mathcal{D}_{0,0}$ as in section 5.1. Stationary states are the same as in section 5.5. They can be represented as tensor products of diagonal 2×2 matrices \mathbf{M} with ω . Symmetry of stationary states includes reflection, exchanging ρ_1 and ρ_2 . Invariant observables are given by all linear combinations of the two projectors onto $\mathbb{C}_{0,j}^2$. They form an algebra.

5.7. Stationary phase relation

The transition operators h_{\pm} are as above in section 5.6, but \mathcal{D} is given without the Hamiltonian. The group of symmetry operations is very rich; there exists many more symmetry operations than in the example of section 5.5. There are more stationary states than in section 5.5, they can be represented as tensor products $\mathbf{M} \otimes \omega$ with any 2×2 matrix $\mathbf{M} > 0$, $\text{Tr}[\mathbf{M}] = 1$. Invariant observables are $\mathbf{A} \otimes \mathbb{1}$; they form an algebra.

5.8. Cascade of decay

$$\mathcal{H} = \mathbb{C}^4 = \mathbb{C}_0 \oplus \mathbb{C}_{1,1} \oplus \mathbb{C}_{1,2} \oplus \mathbb{C}_2$$

$$h_{\pm} = \begin{pmatrix} \cdot & 1 & \pm 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

There are two lines of flow: $|1, 1\rangle\langle 1, 1| \rightarrow |0\rangle\langle 0|$ and $|2\rangle\langle 2| \rightarrow |1, 2\rangle\langle 1, 2| \rightarrow |0\rangle\langle 0|$. Both lines have the same end. The differential equations are $\dot{r}_{1,1} = r_{2,2} + 2r_{3,3}\dot{r}_{2,2} = -r_{2,2}$, $\dot{r}_{3,3} = -2r_{3,3} + r_{4,4}$, $\dot{r}_{4,4} = -r_{4,4}$ for the diagonal matrix elements. The off-diagonals are just exponentially decaying, with one accompanying part of the flow, $\dot{r}_{1,3} = -2r_{1,3} + r_{2,4}$ and the same for the adjoint. There are no invariant observables. See figure 2.

5.9. Maximal symmetric evolution

$\mathcal{H} = \mathbb{C}^n$; the evolution is

$$\dot{\rho} = \dim(\mathcal{H}) \cdot (\omega - \rho).$$

\mathcal{D} can be defined with $H = 0$, $\{h_{i,j} = |i\rangle\langle j|\}$ for some basis $\{|i\rangle\}$. There is a unique stationary state, it is $\omega = \mathbb{1} / \dim(\mathcal{H})$. Symmetry under every unitary and anti-unitary transformation holds. Another way to represent this evolution is to choose the Weyl operators as $\{h_{i,j}\}$ [BA08].

This is a special case of *detailed balance* at infinite temperature (see the remark around equation (2.15) in [G78] and references therein), which appears in all those cases, where

Perturbation case 6.2.5

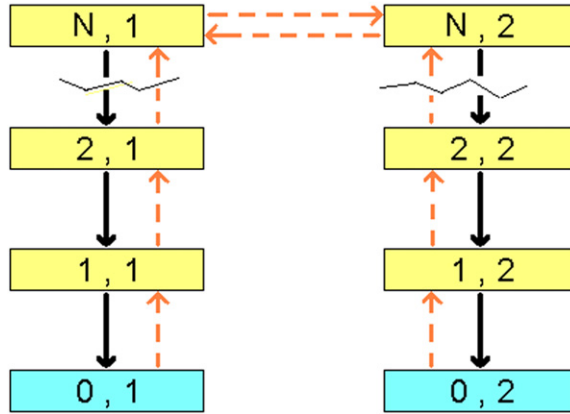


Figure 2. A perturbed cascade. Two independently decaying systems are perturbed in a way as is indicated by the broken pointed lines in red.

$\{h_\alpha^\dagger\} = \{h_\alpha\}$. (Each h_α either has a dual companion $h_\beta = h_\alpha^\dagger$, or it is self-adjoint $h_\alpha^\dagger = h_\alpha$.) One consequence of these symmetries of detailed balance is the invariance of the completely mixed state $\omega = \mathbb{1}/\dim(\mathcal{H})$. Another, related, consequence is the self-adjointness of the superoperator \mathcal{D} , if $H = 0$. Examples 5.1, 5.6 and 5.7 are cases of detailed balance.

6. Perturbations

6.1. Abstract considerations

We consider generators \mathcal{D} depending on a parameter λ . The dependence on λ of the transition operators is assumed as linear, which makes a quadratic dependence on λ for \mathcal{D}_λ . We set $H_\lambda = H_0 + \lambda V + \lambda^2 W$, and $h_{\alpha,\lambda} = h_\alpha + \lambda k_\alpha$. This gives $\mathcal{D}_\lambda = \mathcal{D}_0 + \lambda \mathcal{E} + \lambda^2 \mathcal{F}$ with

$$\mathcal{D}_0(\rho) = -i[H, \rho] + \sum_\alpha \left\{ h_\alpha \rho h_\alpha^\dagger - \frac{1}{2} (h_\alpha^\dagger h_\alpha \rho + \rho h_\alpha^\dagger h_\alpha) \right\}, \tag{41}$$

$$\mathcal{E}(\rho) = -i[V, \rho] + \sum_\alpha \left\{ h_\alpha \rho k_\alpha^\dagger + k_\alpha \rho h_\alpha^\dagger - \frac{1}{2} (h_\alpha^\dagger k_\alpha \rho + \rho h_\alpha^\dagger k_\alpha + k_\alpha^\dagger h_\alpha \rho + \rho k_\alpha^\dagger h_\alpha) \right\}, \tag{42}$$

$$\mathcal{F}(\rho) = -i[W, \rho] + \sum_\alpha \left\{ k_\alpha \rho k_\alpha^\dagger - \frac{1}{2} (k_\alpha^\dagger k_\alpha \rho + \rho k_\alpha^\dagger k_\alpha) \right\}. \tag{43}$$

This formalism includes cases where the set of transition operators is enlarged. Formally, it is done by perturbing some $h_\alpha = 0$ with $k_\alpha \neq 0$.

A general fact is that the defining equations for *structuring*, i.e. (22) for ‘laziness’ and (24) for ‘nothing goes out’, and for *symmetry*, i.e. (39), can abruptly turn to inequalities through an infinitesimal change $\lambda \rightarrow \lambda + d\lambda$, but not the other way round:

Proposition 19. *Consider the projection operators $P(\lambda)$ onto subspaces, which are either enclosures, or collecting subspaces, or basins in the cascade of decay. Consider also the unitary and antiunitary operators $V(\lambda)$ which are symmetry operations as defined in section 4.2.*

The functions $\lambda \mapsto P(\lambda)$ and $\lambda \mapsto V(\lambda)$ are continuous functions, defined on closed sets of $\lambda \in \mathbb{R}$.

Proof. To study ‘laziness’ under the influence of perturbations $h_\alpha \mapsto h_\alpha(\lambda)$ define $f(P, \lambda) := \sum_\alpha \|h_\alpha(\lambda)P - Ph_\alpha(\lambda)P\|$ for orthogonal projectors P . The projectors P can be expressed by a finite set of parameters, e.g. its matrix elements. The function $f(P, \lambda)$ is jointly continuous in P and λ , so the set $\{(\lambda, P) | f(P, \lambda) = 0, P = P^2 = P^\dagger\}$, which defines lazy subspaces via the implicit functions $P(\lambda)$, is closed. It is compact, when λ is restricted to a compact interval. The implicit functions $P(\lambda)$ can be multi-valued; they give minimal lazy subspaces and direct sums of them. The domain of $\lambda \in \mathbb{R}$ for one $P(\lambda)$ is closed, the projector may disappear under infinitesimal changes of λ .

To study ‘collecting’ subspaces, consider the functions $f_C(P, \lambda) := \|P(iH(\lambda) - \frac{1}{2} \sum_\alpha h_\alpha^\dagger(\lambda)h_\alpha(\lambda))P^\perp\|$, and proceed in the same way as for laziness. To study symmetry, use $f_S(V, \lambda) := \|\mathcal{D}_\lambda(V\rho V^{-1}) - V\mathcal{D}_\lambda(\rho)V^{-1}\|$. \square

Infinitesimal changes of the transition operators and of the Hamiltonian can lead to mergers, may disturb an existing structure, or move it in the Hilbert space, but they cannot create a new one. Moving basins in the Hilbert space cannot occur through mere addition of new transition operators, since condition (22) for laziness involves each single h_α .

A list of things that can happen:

- The number of zero eigenvalues may decrease (but it cannot increase). Example: disturb section 5.2 with $k_\pm = \pm|0, 1\rangle\langle 0, 2|$.
- Stationary phase relations may turn to undamped oscillating ones, or they may become unstable. The inverse changes are not possible. Example: disturb section 5.3 with the Hamiltonian $H = \pm|0, 1\rangle\langle 0, 1|$ or with $k_\pm = \pm|0, 1\rangle\langle 0, 1|$.
- Oscillating phase relations may disappear. Example: disturb section 5.6 with the h_\pm of 5.5.
- Enclosures may merge. Example : disturb section 5.5 with $k_+ = |0\rangle\langle 1| \otimes \mathbb{1}$.
- Collecting basins may merge. Example: the same as for the first item.
- Collecting basins may merge with decaying basins. Example: disturb section 5.8 with $k_\pm = \pm|0\rangle\langle 1, 2|$.
- Rotations of basins in the Hilbert space. Example: consider the simple generator with transition operator $h = \sigma_x$, disturb it with $k = \sigma_y$.
- Symmetries may disappear (but new symmetries cannot emerge). Example: below, in section 6.2.1, the worked out perturbation of section 5.1.

The general perturbation theory of linear operators, [Kato], on finite-dimensional spaces states the analyticity of eigenvalues and eigenprojectors onto eigenspaces, with only algebraic singularities at some exceptional points. Now we are interested in the eigenvalue zero. Its multiplicity may decrease under perturbation, but the eigenvalue zero has to remain, with multiplicity one at least. We want to follow this eigenvalue and the remaining eigenprojectors onto the stationary states. The constant function $\lambda \mapsto 0$ is obviously analytic. So we have analyticity of the eigenprojector, from which we pick out the projectors onto those eigenmatrices which are states. Relying on the analyticity at $\lambda = 0$ for

$$\lambda \mapsto \{H_\lambda, h_{\alpha,\lambda}\} \mapsto \mathcal{D}_\lambda \mapsto \{\rho(\lambda)\}$$

we make an ansatz, expanding stationary states:

$$\rho(\lambda) = \rho + \sum_{n=1}^{\infty} \lambda^n \sigma_n. \tag{44}$$

We demand $\text{Tr}[\rho(\lambda)] = 1$, which gives

$$\text{Tr}[\rho] = 1, \quad \text{Tr}[\sigma_n] = 0. \tag{45}$$

Expansion of the eigenvalue equation $\mathcal{D}_\lambda(\rho(\lambda)) = 0$ gives the starting condition

$$\mathcal{D}_0(\rho) = 0 \tag{46}$$

and a series of equations to determine the following contributions:

$$\mathcal{D}_0(\sigma_n) = -\mathcal{E}(\sigma_{n-1}) - \mathcal{F}(\sigma_{n-2}), \tag{47}$$

using $\sigma_{-1} = 0$ and $\sigma_0 = \rho$. Trying to solve one of these equations one encounters two problems: solving it requires that the rhs is a matrix in range $[\mathcal{D}_0]$. Note that the range of all the operators involved in (47) consists of matrices with trace zero. This is part of solution to this first problem. The second problem: an inverse of \mathcal{D}_0 is not uniquely given, if (46) allows for more than one solution, i.e. if more than one stationary state exists. Choosing the right solution appears in the iterated equations as solving there the first problem, deciding whether the rhs of (47) is in the range of \mathcal{D}_0 .

In order to proceed with calculations we define an inverse of \mathcal{D}_0 independently of the other super-operators:

$$\text{range}[\mathcal{D}_0^{-1}] := \text{range}[\mathcal{D}_0] = \{\tau | \forall A_i \text{ which are invariant: } \text{Tr}[A_i \tau] = 0\}, \tag{48}$$

with using a maximal set of linearly independent invariant operators A_i . This makes the operator \mathcal{D}_0^{-1} unique: if there exists σ , such that $\mathcal{D}_0(\sigma) = \tau$, then also $\mathcal{D}_0(\sigma + \sum_j \alpha_j \rho_j) = \tau$, with HS orthogonal eigenmatrices—stationary states and stationary phase relations— ρ_j . Choose the A_i in such a way that $\text{Tr}[A_i \rho_j] = \delta_{i,j}$, then

$$\text{Tr} \left[A_i \left(\sigma + \sum_j \alpha_j \rho_j \right) \right] = \text{Tr}[A_i \sigma] + \alpha_i = 0 \tag{49}$$

determines the α_i uniquely.

With this generally defined inverse of \mathcal{D}_0 one can invert (47) to

$$\sigma_n = -\mathcal{D}_0^{-1}(\mathcal{E}(\sigma_{n-1}) + \mathcal{F}(\sigma_{n-2})) + \sum_j \alpha_j \rho_j, \quad \text{with} \quad \sum_j \alpha_j \text{Tr}[\rho_j] = 0, \tag{50}$$

and the task is now to find the right coefficients α_j , so that the insertion of σ_n into the next iteration formulae gives matrices in the domain of definition of \mathcal{D}_0^{-1} —which is the range of \mathcal{D}_0 . Here we refrain from establishing a complete formalism, we represent several case studies instead, including special formulae.

6.2. Case studies and examples

6.2.1. Only one stationary state for the unperturbed system. If there is only one ρ as eigenmatrix to the zero-eigenvalue of \mathcal{D}_0 , there are no more tasks to fulfil than to perform the calculations. It is not necessary to make a difference between systems with decay and those without. One may define superoperators \mathcal{G}_n as

$$\mathcal{G}_0 = \mathbb{1}, \quad \mathcal{G}_1 = -\mathcal{D}_0^{-1} \circ \mathcal{E}, \quad \mathcal{G}_n = -\mathcal{D}_0^{-1} \circ (\mathcal{E} \circ \mathcal{G}_{n-1} + \mathcal{F} \circ \mathcal{G}_{n-2})$$

and use them to calculate

$$\rho(\lambda) = \rho + \sum_n \lambda^n \mathcal{G}_n(\rho). \tag{51}$$

As an example to demonstrate the validity of this procedure we disturb section 5.1 with $k_+ = (\mathbb{1} - \sigma_z)/2$, leaving h_- undisturbed. This gives $\sigma_{2n} = 0$ and $\sigma_{2n+1} = (-2)^{-n} \sigma_x/2$. The series converges for $|\lambda| < \sqrt{2}$ and can be summed up, giving

$$\rho(\lambda) = \omega + \frac{\lambda}{1 + \lambda^2/2} \sigma_x/2. \tag{52}$$

6.2.2. Remaining enclosures, each one with only one stationary state. Consider the case of projectors Q_j commuting with $H_\alpha(\lambda)$ and with every $h_\alpha(\lambda)$. Inside each enclosure $Q_j \mathcal{H}$ the situation is the same as above, in section 6.2.1. If there exist enclosures which allow pairwise stationary phase relations, collect them as $\mathbb{C}^n \otimes \mathcal{H}$. The evolution takes place only in \mathcal{H} and, if also the perturbation shows this symmetry, it can again be treated as in section 6.2.1. If it disturbs this symmetry, the phase relations vanish by dephasing.

We remark that besides the families of stationary states $\sum_j \beta_j \rho_j(\lambda)$ found in this way there could, formally, also be families with the β_j depending on λ .

6.2.3. Merging of enclosures through direct dissipation. Consider $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where each $\mathcal{H}_j = Q_j \mathcal{H}$ is an enclosure supporting just one stationary state ρ_j , and where no stationary phase relations exist. Perturb a (virtual) $h_\alpha = 0$ with k , so the \mathcal{E} in (42) is zero. The perturbing transition operator connects the enclosures via $Q_1 k Q_2 \neq 0$, the other off-diagonal block of the matrix k may be zero or not. We demand moreover

$$Q_1 k \rho_2 \neq 0. \tag{53}$$

The starting condition $\rho = \alpha \rho_1 + (1 - \alpha) \rho_2$, and the first of conditions (47), which is $\mathcal{F}(\rho) \in \text{range}(\mathcal{D}_0)$, here to be expressed as $\text{Tr}[Q_1 \mathcal{F}(\rho)] = 0$, give, using $Q_1 = \mathbb{1} - Q_2$ and $\text{Tr}[\mathcal{F}(\rho_1)] = 0$,

$$\begin{aligned} &\alpha \text{Tr}[Q_1 \mathcal{F}(\rho_1)] + (1 - \alpha) \text{Tr}[Q_1 \mathcal{F}(\rho_2)] \\ &= \text{Tr}[Q_1 k \rho_2 k^\dagger Q_1] - \alpha (\text{Tr}[Q_2 k \rho_1 k^\dagger Q_2] + \text{Tr}[Q_1 k \rho_2 k^\dagger Q_1]) = 0. \end{aligned} \tag{54}$$

That determines $\alpha \in (0, 1]$ uniquely, since (53) implies $\text{Tr}[Q_1 k \rho_2 k^\dagger Q_1] > 0$.

To solve the following conditions of (47),

$$\mathcal{F}(\sigma_{2n+2}) = -\mathcal{F}(\mathcal{D}_0^{-1}(\mathcal{F}(\sigma_{2n}))) + \alpha \cdot \mathcal{F}(\rho_1 - \rho_2) \in \text{range}(\mathcal{D}_0),$$

we define the functional

$$\alpha[\sigma] = \frac{\text{Tr}[Q_1 \cdot \mathcal{F}(\mathcal{D}_0^{-1}(\mathcal{F}(\sigma)))]}{\text{Tr}[Q_2 k \rho_1 k^\dagger Q_2] + \text{Tr}[Q_1 k \rho_2 k^\dagger Q_1]}.$$

The sequence σ_n is now fixed as $\sigma_{2n+1} = 0$ and

$$\sigma_{2n+2} = -\mathcal{D}_0^{-1}(\mathcal{F}(\sigma_{2n})) + (\rho_1 - \rho_2) \cdot \alpha[\sigma_{2n}].$$

6.2.4. Dephasing perturbed by a Hamiltonian. Consider again $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where each $\mathcal{H}_j = Q_j \mathcal{H}$ is an enclosure supporting just one stationary state ρ_j , and where no stationary phase relations exist. Perturb \mathcal{D}_0 with a Hamiltonian V , so the \mathcal{F} in (43) is zero. The perturbing V connects the enclosures via $Q_1 V Q_2$ and $Q_2 V Q_1$; we demand

$$Q_1 V \rho_2 \neq 0. \tag{55}$$

For simplicity, assume $Q_1 V Q_1 = 0$ and also $Q_2 V Q_2 = 0$.

Condition (55) implies, as can be checked by considering matrix elements involving eigenvectors of the ρ_j ,

$$[V, \rho_1 - \rho_2] \neq 0.$$

Since \mathcal{D}_0 does not mix the matrix blocks $Q_j \sigma Q_k$ and does not annihilate phase relations, the range of \mathcal{D}_0 contains every $Q_1 \sigma Q_2$, every $Q_2 \sigma Q_1$, and the inverse \mathcal{D}_0^{-1} does exist for every off-diagonal block. This applies to $\mathcal{E}(\rho)$, where $\rho = \alpha_0 \rho_1 + (1 - \alpha_0) \rho_2$. So the first order of perturbation theory gives no restriction on α_0 . We proceed with

$$\sigma_1 = -\mathcal{D}_0^{-1}(\mathcal{E}(\rho_2)) - \alpha_0 \mathcal{D}_0^{-1}(\mathcal{E}(\rho_1 - \rho_2)) + \alpha_1 (\rho_1 - \rho_2).$$

The condition $\mathcal{E}(\sigma_1) \in \text{range}(\mathcal{D}_0)$ does not involve α_1 but it determines the right value for α_0 , by demanding

$$\alpha_0 \text{Tr}[Q_1 \mathcal{E}(\mathcal{D}_0^{-1}(\mathcal{E}(\rho_1 - \rho_2)))] = -\text{Tr}[Q_1 \mathcal{E}(\mathcal{D}_0^{-1}(\mathcal{E}(\rho_2)))],$$

in case $\text{Tr}[Q_1 \cdot \mathcal{E}(\mathcal{D}_0^{-1}(\mathcal{E}(\rho_1 - \rho_2)))] \neq 0$. The abstract considerations of section 6.1 imply that this factor has to be non-negative. In fact, the part due to $-\rho_2$ may, with some tricky methods, be written as $\sum_{\alpha} \text{Tr}[A_{\alpha} A_{\alpha}^{\dagger}]$, with $A_{\alpha} = h_{\alpha} \mathcal{D}_0^{-1}(V \rho_2) \rho_2^{-1/2} P_{\rho} - \mathcal{D}_0^{-1}(V \rho_2) \rho_2^{-1} P_{\rho} h_{\alpha} \rho_2^{1/2}$, where P_{ρ} is the projector onto the range of ρ , and the h_{α} are the transfer operators appearing in \mathcal{D}_0 . The part of the factor involving ρ_1 is zero if $\rho_1 V = 0$, but in general it is also non-negative. We assume now, that at least one $A_{\alpha} \neq 0$, or, for short, just that the factor of α_0 , which appears also as factor for all the following α_n , is not zero.

The expansion of $\rho(\lambda)$ proceeds with

$$\sigma_{n+1} = \mathcal{D}_0^{-1}(\mathcal{E}(\mathcal{D}_0^{-1}(\mathcal{E}(\sigma_{n-1})))) - \alpha_n \mathcal{D}_0^{-1}(\mathcal{E}(\rho_1 - \rho_2)) + \alpha_{n+1} (\rho_1 - \rho_2).$$

The condition $\mathcal{E}(\sigma_{n+1}) \in \text{range}(\mathcal{D}_0)$ can be fulfilled by choosing $\alpha_n = \alpha[\sigma_{n-1}]$, with the functional

$$\alpha[\sigma] = \frac{-\text{Tr}[Q_1 \cdot \mathcal{E}(\mathcal{D}_0^{-1}(\mathcal{E}(\mathcal{D}_0^{-1}(\mathcal{E}(\sigma)))))]}{\text{Tr}[Q_1 \cdot \mathcal{E}(\mathcal{D}_0^{-1}(\mathcal{E}(\rho_1 - \rho_2)))]}.$$

6.2.5. Basins of a cascade which merge by dissipation. Consider $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where each $\mathcal{H}_j = Q_j \mathcal{H}$ is spanned by the $N + 1$ lowest levels $\psi_{j,0}, \dots, \psi_{j,N}$ of a Harmonic oscillator. The decay is performed through the annihilation operators a_j acting as transition operators. This system is now perturbed by adding \mathcal{F} made of three new \mathcal{D} . Two of them involve the creation operators a_j^{\dagger} , the third one creates a dissipation between the two levels $\psi_{j,N}$. Finding the right constants α_{2n} to do the expansion

$$\sigma_{2n+2} = -\mathcal{D}_0^{-1}(\mathcal{F}(\sigma_{2n})) + \alpha_{2n+2} \cdot (\rho_1 - \rho_2)$$

requires now an N -fold iteration of applying the super-operator $\mathcal{G} := -\mathcal{D}_0^{-1} \circ \mathcal{F}$. It takes N steps of applying \mathcal{G} until the unperturbed ground states ρ_j are lifted to the N th level, and one more action of \mathcal{F} to dissipate over to the other \mathcal{H}_j . The procedure to get $\alpha_{2n+2} = \alpha[\sigma_{2n}]$ now involves the functional

$$\alpha[\sigma] = \frac{\text{Tr}[Q_1 \cdot (\mathcal{F} \circ \mathcal{G}^{N+1})(\sigma)]}{\text{Tr}[Q_1 \cdot (\mathcal{F} \circ \mathcal{G}^N)(\rho_1 - \rho_2)]}.$$

7. Summary

Analysis of Lindblad generators is to be done on two levels. The lower level is the finite-dimensional Hilbert space \mathcal{H} spanned by the pure-state vectors. The generators and their processes act at the upper level which contains the mixed states represented by density matrices. In this paper we have established an interplay of these two levels, where every process corresponds to a structure of \mathcal{H} , a decomposition into mutual orthogonal subspaces. These subspaces, we call them basins, generalize the notion of energy levels which appears in Hamiltonian dynamics. Their dimensions may, in general, be any number between 1 and the dimension of the whole Hilbert space. They are mutually orthogonal, in spite of the possible non-Hermiticity of the super-operators which represent the Lindblad generators and of the transfer operators acting on \mathcal{H} .

The interplay between process and Hilbert space structure elucidates the characterization of the process, and brings about a way to structure the process itself, according to its action on blocks of density matrices. *Decay* corresponds to certain subspaces which do not carry any stationary state. Minimal stationary states are supported by minimal collecting basins, inside of which *dissipation* occurs. The phase relations between two basins, off-diagonal blocks of density matrices, either show *dephasing*, or, in course of the process, go over to a special phase relation, which is either stationary or oscillating.

Stationary and oscillating phase relations appear together with special dynamical *symmetries*. The appearances of symmetries and of invariant observables may show peculiar effects. In cases of decay the invariant observables need not form an algebra. Dynamical symmetries are not the same as the symmetries appearing for the set of stationary states, in general. *Perturbation* of a process may lead to a merging of basins, which complicates the perturbative calculations of stationary states.

Establishing the structure connected with a process and its Lindblad generator is, we think, a helpful tool for deeper investigations, probably indispensable.

Appendix

A.1. block matrix form of operators to characterize subspaces

The subspace which is to be characterized, is represented with the projector $P = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$. It carries the density matrices $\rho = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$.

The subspace $P\mathcal{H}$ has the property of being Lazy $\Leftrightarrow h_\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ 0 & D_\alpha \end{pmatrix}$. Collecting \Leftrightarrow Lazy and, with $H = \begin{pmatrix} H_P & G \\ G^\dagger & L \end{pmatrix}$, $iG - \frac{1}{2} \sum_\alpha A_\alpha^\dagger B_\alpha = 0$. An enclosure $\Leftrightarrow H = \begin{pmatrix} H_P & 0 \\ 0 & L \end{pmatrix}$, $h_\alpha = \begin{pmatrix} A_\alpha & 0 \\ 0 & D_\alpha \end{pmatrix}$.

See also [BNT08], where formulae for calculating $\mathcal{D}_h(\rho)$ in the block-matrix form are presented.

References

- [L76] Lindblad G 1976 On the generators of quantum dynamical semigroups *Commun. Math. Phys.* **48** 119–30
- [GKS76] Gorini V, Kossakowski A and Sudarshan E C G 1976 Completely positive dynamical semigroups of N -level systems *J. Math. Phys.* **17** 821–5
- [D76] Davies E B 1976 *Quantum Theory of Open Systems* (New York: Academic)
- [S76] Spohn H 1976 Approach to equilibrium for completely positive dynamical semigroups of N -level systems *Rep. Math. Phys.* **10** 189–94
- [F78] Frigerio A 1978 Stationary states of quantum dynamical semigroups *Commun. Math. Phys.* **63** 269–76

- [S80] Spohn H 1980 Kinetic equations from Hamiltonian dynamics *Rev. Mod. Phys.* **52** 569–615
- [BNT08] Baumgartner B, Narnhofer H and Thirring W 2008 Analysis of quantum semigroups with GKS–Lindblad generators: I. Simple generators *J. Phys. A: Math. Theor.* **41** 065201
- [J03] Joos E *et al* 2003 *Decoherence and the Appearance of a Classical World* 2nd edn (Berlin: Springer)
- [Z03a] Zurek W H 2003 Decoherence and the transition from quantum to classical (updated) arXiv:quant-ph/0306072 (an updated version of 1991 *Physics Today* **44** 3644)
- [Z03b] Zurek W H 2003 Decoherence, einselection, and the quantum origins of the classical *Rev. Mod. Phys.* **75** 715 (Preprint [quant-ph/0105127v3](http://arxiv.org/abs/quant-ph/0105127v3))
- [AF01] Alicki R and Fannes M 2001 *Quantum Dynamical Systems* (Oxford: Oxford University Press)
- [BP02] Breuer H-P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
- [J03b] Jones V F R 2003 Von Neumann algebras, incomplete notes from a course <http://math.berkeley.edu/~vfr/VonNeumann.pdf>
- [L69] Lancaster P 1969 *Theory of Matrices* (New York: Academic)
- [K08a] Kraus B *et al* 2008 Preparation of entangled states by dissipative quantum Markov processes Preprint [arXiv:0803.1463](http://arxiv.org/abs/0803.1463)
- [K08b] Diehl S *et al* 2008 Quantum states and phases in driven open quantum systems with cold atoms Preprint [arXiv:0803.1482](http://arxiv.org/abs/0803.1482)
- [G78] Gorini V *et al* 1978 Properties of quantum Markovian master equations *Rep. Math. Phys.* **13** 149–73
- [BA08] Ableitinger Alexander 2008 Decoherence and open quantum systems *Diplomarbeit unter Anleitung von Reinhold Bertlmann* (Universität Wien)
- [Kato] Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin: Springer)